J is thus the determinant of the Jacobian of the transformation, or the "functional determinant."

The strain, N_{jk}, is defined, somewhat arbitrarily, from the difference in the squares of the lengths of line elements by:

$$2N_{jk}\, da_j\, da_k = dx_i\, dx_i - da_i\, da_i$$

$$N_{jk} = 1/2\left(\frac{\partial x_i}{\partial a_j}\frac{\partial x_i}{\partial a_k} - \delta_{jk}\right) \tag{2.15}$$

Here and in the following the Einstein summation convention for repeated subscripts applies. δ_{jk} is the Kronecker delta.

Expanding the internal (strain) energy in a power series in the strains, one obtains (at constant entropy):

$$\rho_o[E(N,S) - E(0,S)] = 1/2\, c^S_{ijkl}\, N_{ij}\, N_{kl} + 1/6\, c_{ijklmn}\, N_{ij}\, N_{kl}\, N_{mn}$$

$$+ 1/24\, c^S_{ijklmnpq}\, N_{ij}\, N_{kl}\, N_{mn}\, N_{pq} + \ldots \tag{2.16}$$

In this expression the c^S_{ijk} . . . , represent the second and higher order isentropic elastic stiffness coefficients. The first-order term is missing since the reference state is considered to be one of zero stress and strain.

We now define quantities, called thermodynamic tensions, by

$$t_{ij} = \rho_o\left(\frac{\partial E}{\partial N_{ij}}\right)_s \quad . \tag{2.17}$$

In terms of these quantities the elastic constants are

$$c^S_{ijkl} = \left(\frac{\partial t_{ij}}{\partial N_{kl}}\right)_s = \frac{\partial^2 E}{\partial N_{ij}\, \partial N_{kl}}$$

and similarly for the higher order coefficients. Consequently,

$$\rho_o \, dE = t_{ij} \, dN_{ij} \; (dS = 0) \quad .$$

Finally, the equilibrium (non-dissipative) components of the stress are obtained from the thermodynamic tensions by the relations,

$$\sigma_{km} = (1/j) \frac{\partial x_k}{\partial a_j} \frac{\partial x_m}{\partial a_i} t_{ij} \quad . \qquad (2.18)$$

The above formulas provide isentropic constitutive relations in terms of the elastic stiffness coefficients. Other forms of constitutive relations can, of course, be derived in a similar fashion.

Low pressure acoustic measurements yield a mixed third-order constant of the form:

$$C_{ijkmpq} = \left(\frac{\partial c^S_{ijkm}}{\partial N_{pq}}\right)_T$$

where the subscript T means the derivative is taken at constant temperature. The corresponding purely isentropic constant is given by:

$$c^S_{ijkmpq} = C_{ijkmpq} + (T/\rho_o \, C_t) \, c^S_{kmpq} \, \alpha_{uv} [C_{ijkmrs} \, \alpha_{rs} - \left(\frac{\partial c^S_{ijkm}}{\partial T}\right)_t] \qquad (2.19)$$

where C_t is the specific heat at constant tension and the α_{uv} are thermal expansion coefficients,

$$\alpha_{uv} = \left(\frac{\partial N_{uv}}{\partial T}\right)_t \quad .$$

In view of the symmetry of the stress and strain tensors, the number of subscripts can be reduced by adopting the following convention:

$$11 \rightarrow 1 \qquad 32 \rightarrow 4$$
$$22 \rightarrow 2 \qquad 31 \rightarrow 5$$
$$33 \rightarrow 3 \qquad 21 \rightarrow 6$$

This convention is employed in the following.